Applied Soft Computing 15 (2014) 193–202

Contents lists available at ScienceDirect

Applied Soft Computing

journal homepage: www.elsevier.com/locate/asoc

Linear and sigmoidal fuzzy cognitive maps: An analysis of fixed points

Christopher J.K. Knight [a,b,*], David J.B. Lloyd [b], Alexandra S. Penn [a,c]

[a] ERIE, Department of Sociology, University of Surrey, Guildford GU2 7XH, UK
[b] Department of Mathematics, University of Surrey, Guildford GU2 7XH, UK
[c] Center for Environmental Strategy, University of Surrey, Guildford GU2 7XH, UK

ARTICLE INFO

Article history:
Received 13 May 2013
Received in revised form 2 October 2013
Accepted 29 October 2013
Available online 12 November 2013

Keywords:
Fuzzy cognitive maps
Fixed points
Stability
Nonlinear dynamical system

ABSTRACT

Fuzzy cognitive mapping is commonly used as a participatory modelling technique whereby stakeholders create a semi-quantitative model of a system of interest. This model is often turned into an iterative map, which should (ideally) have a unique stable fixed point. Several methods of doing this have been used in the literature but little attention has been paid to differences in output such different approaches produce, or whether there is indeed a unique stable fixed point. In this paper, we seek to highlight and address some of these issues. In particular we state conditions under which the ordering of the variables at stable fixed points of the linear fuzzy cognitive map (iterated to) is unique. Also, we state a condition (and an explicit bound on a parameter) under which a sigmoidal fuzzy cognitive map is guaranteed to have a unique fixed point, which is stable. These generic results suggest ways to refine the methodology of fuzzy cognitive mapping. We highlight how they were used in an ongoing case study of the shift towards a bio-based economy in the Humber region of the UK.

1. Introduction

Fuzzy cognitive mapping is a method for representing a system based on expert knowledge as a directed weighted graph, with weights chosen from a set, resulting in a semi-quantitative model of the system. This method was first developed by Kosko [15] to better understand systems with numerous interconnections between important components but relatively scarce quantifiable information about the form of these interconnections.

Fuzzy cognitive maps (FCMs) have since been used to model a huge variety of systems, from a heat exchanger [26] to predicting prostate cancer [7]. However the way they are used has developed differently amongst different communities of researchers. These separate development paths can loosely be associated with the community of engineers and biologists, and the community of social scientists. In the former it is possible to know what the system should look like on some level, and thus to provide some quantitative data to fit the model to. This allows learning algorithms to be used in order to improve the model [7,16,12,21,22]. It may also be possible to provide direct feedback from the model into the system it represents to allow the model to be used to directly control the system [16,26]. This has been studied more for general fuzzy models than for FCMs. For instance with appropriate learning algorithms fuzzy models can be used in stabilising an inverted pendulum [10] or in controlling the effect of earthquakes on structures [4,30]. In the social sciences however such direct relationships between the model and the system it represents are not possible, meaning that learning algorithms are rarely applied. In fact even representing the system has its complexities as any representation will be a subjective interpretation of the system. In these disciplines FCMs have developed as an explanatory tool, as a relatively simple way of representing a complex system. From such a model a narrative based analysis can be used to explain how the system might react to certain changes [11,14,18,23,28]. Another way in which the use of FCMs has developed in such subjective fields has been to incorporate the knowledge and insight of those who are enmeshed in the system. That is, the FCM has developed into a participatory modelling methodology where the intersubjective knowledge of stakeholders is harnessed to create a model of the system they inhabit. In this paper we focus on this aspect of fuzzy cognitive mapping, however the results are applicable to other uses of FCMs as well.

Within the social sciences participatory fuzzy cognitive mapping is used across a wide range of disciplines, such as environmental management, organisational management, urban design and industrial ecology [3,5,6,9,11,20,23,28,29]. For instance, it has recently been used to better understand such disperse systems as land cover in the Brazilian Amazon [24], the future of water in the Seyhan Basin [3], and as a method for using stakeholders in the product development process [11]. This methodology enables the creation of semi-quantitative models representing the knowledge of 'on-the-ground' experts in the system. These models can then be analysed by the researcher to gain insight into the beliefs inherent in the system, and by the participants to develop their

* Corresponding author. Tel.: +44 1483 683617.
E-mail address: c.j.knight@surrey.ac.uk (C.J.K. Knight).

1568-4946/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.asoc.2013.10.030

own understanding of the system and how all its constituent parts interact.

Participatory fuzzy cognitive mapping generally consists of a group exercise in which the stakeholders collaborate to create a semi-quantitative model of the system. This model takes the form of a list of key concepts and a list of weighted causal links between these concepts. First of all the key concepts are discussed and agreed upon. Then causal links are discussed, along the lines of: if concept x increased or decreased would that have a *direct* effect on concept y. This creates a directed network (known as a cognitive map [15]). The final stage of the initial modelling is to assign 'fuzzy' weights, that is values from a closed set of weights, to the edges of the directed graph. The specific way that this is done varies from case to case. For example, it could be done by (i) assessing whether the link is positive (an increase in x causes an increase in y) or negative (an increase in x causes a decrease in y), or (ii) by assessing whether the link is positive or negative and whether it is weak, medium or strong [1,23,24]. As an example of this participatory methodology in practice, in a study of bio-based energy production in the Humber region [23] participants identified 16 key concepts with 27 links between them, the participants then assessed the direction and strength of all of these links. For instance, it was reasoned that any increase in bio-based energy production would lead to an increase in the number of jobs and that this causal effect was of medium strength (the weights were determined by ranking the importance of the links). Thus, that edge was labelled positive-medium.

The qualitative weights are assigned numerical values to form a quantitative model (the fuzzy cognitive map (FCM)). Several different choices are made in the literature for the assigning of such numerical values, typically lying in the interval $[-1, 1]$ [1,13,23,24], for instance [24] uses weights in the set $[-0.7, -0.5, -0.2, 0, 0.2, 0.5, 0.7]$ for links which are negative or positive and strong, medium or weak.

The process of creating a model with stakeholders is often worthwhile in and of itself for its engagement value, and for the insights offered to the researcher by stakeholders knowledgeable in the 'real' behaviour of the system. However, the use of FCMs does not stop there, the weighted network of causal relations is often turned into an iterative map [9,14,20,21,24] which is used to update values assigned to the key concepts. This requires some initial estimate of the values of the concepts. As the concepts are often all widely different and may be subjective, it is hard to assign them values, however this issue can be resolved by normalising to some 'best' case and some 'worst' case value, meaning that the values of the concepts are bounded, typically to the interval $[-1, 1]$ or $[0, 1]$. The iterative process is given by the mapping

$$\underline{x}_{n+1} = f(A\underline{x}_n), \tag{1}$$

where $\underline{x}_n$ is a vector consisting of the values of the concepts at 'time' n, and A is the directed, weighted adjacency matrix. Various monotonic functions f have been taken in the literature, such as step functions, sigmoidal functions, ramp functions and linear functions [9,14,20,21,24], sometimes with slightly different implementation procedures.

In [9] it is suggested that in analysing the map (1) one should focus on the fixed points, as for long term policy decisions the initial transient dynamics are not of interest. Furthermore since 'time' is not defined in the modelling process it is very hard to translate and interpret temporal dynamics. Focussing on fixed points means that one of the most important questions concerns the uniqueness of stable fixed points. This question shall form the main focus of this paper, and in particular we shall find conditions which guarantee the uniqueness of a fixed point for both linear and sigmoidal FCMs. This is particularly important in such a 'fuzzy' participatory setting and often appears to be implicitly assumed. If there were two stable fixed points then a slight change in initial conditions (which cannot be quantified exactly) may result in a totally different outcome, making the value of the map hard to justify. Some preliminary work has already been done seeking to address this question. For instance [9], citing [19], shows that if f is a three region piecewise linear ramp function (between 0 and 1) then (1) has a unique fixed point. Another example of work in this area is [17, Theorem 4] which gives a tight bound on when a sigmoidal FCM has a unique fixed point for a given adjacency matrix.

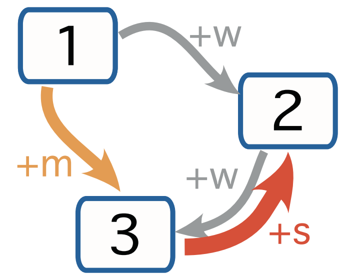

Fig. 1. A simple qualitatively weighted directed network.

Thus it seems sensible to study the existence and stability of solutions to (1) in some generality. This is what we do in this paper. Primarily we focus on general linear and sigmoidal maps and prove results about the existence, uniqueness and stability of fixed points in such systems. For instance we present a similar result to that in [17] for sigmoid FCMs, we give a generic weak bound when the adjacency matrix is not known a priori (when creating an FCM it would be beneficial to know beforehand that there will be a unique fixed point). The proof of this result is simpler than that of the similar result in [17].

It also appears that little thought has been given to the way in which the function used affects the behaviour of the iterative map, and in particular how it affects the existence and stability of fixed points, and the ordering of the concepts there. Exceptions are [27] which compares the output of three different types of FCM and [2] which compares the output of four different types of FCM. So before concentrating on the existence and stability of fixed points we first present a simple example which shows how drastically the function used, f, can affect the ordering of the concepts at the fixed point.

1.1. Example

We present a simple example with three concepts (see Fig. 1) that highlights some of the difficulties in implementing and interpreting FCMs with different functional forms f. We shall use three functional forms for f: linear, sigmoidal and step.

We use the weights 0.3 for a weak link, 0.5 for a medium strength link and 0.8 for a strong causal link. This means that we can express all the information contained in the network via (the transpose of) its adjacency matrix:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0.3 & 0 & 0.8 \\ 0.5 & 0.3 & 0 \end{pmatrix}.$$

With this matrix we can iterate the map (1) with different functions f. However as concept 1 has no input its value will immediately take on the value $f(0)$ and remain at that value thereafter. If this value is zero (it typically is in the linear case) then this will cause all concept values to tend to zero. To avoid this scenario such a concept is generally made a 'driver' of the system [14,24] and given a self reinforcing loop of weight 1 (generally, a concept which has a self reinforcing loop is referred to as a driver). We do this here, setting $A_{1,1} = 1$. This represents the fact that the value of concept 1 does not depend on the values of the other concepts. As an initial condition we take $x_0 = (1, 0, 0)$, the driver has value 1 and all other concepts, value 0. We can then iterate the map (1) for any function f. Below we use a linear function, a step function and a sigmoid function.

However when using a sigmoidal FCM there is a competing procedure for determining drivers [8,21,26]. Instead of using the matrix A in the iterative map, the matrix $A + I$ is used, where I is the identity, or equivalently, all concepts are made drivers. This methodology views the combined effects of the links as an increase or decrease to the value of the concept, rather than the value of the concept itself. To be consistent we use the same initial condition for this procedure, that is $x_0 = (1, 0, 0)$. In Fig. 2 we show the forward iterations of (1) converging to a fixed point for: a linear function ($f(x) = x$); a step function ($f(x) = 0$ for $x < h$ and $f(x) = 1$ for $x > h$, with $h = 0.5$); and a sigmoid function ($f(x) = 1/(1 + e^{-\lambda(x-h)})$ with $h = 0.5$ and $\lambda = 2$). We show both possible implementations of the sigmoid function; when all concepts are drivers and when only concept 1 is a driver.

In interpreting the values of the concepts at the fixed point we must remember the fuzzy nature of the original model. This means that only the ordering of the concepts at the fixed point is important, and not their specific value. From this simple example we can make the observation that the function used in the FCM and the implementation procedure can vastly affect the orderings of the concepts and thus the interpretation of the model (a fact also noted in [27] for different functional forms, and discussed in the bio-based energy in the Humber region case study, [23]). In fact for this simple three concept example the ordering is different in each of the four cases we implemented. Whilst this result is unsurprising it is nevertheless important, particularly in regard to the scope of the differences. As the output (the ordering) is so dependent on the function used and the implementation procedure, careful justification needs to be made in these choices. Simply choosing the linear FCM, for example, because it is simplest to implement [14,28], risks the output from the fixed point, or any scenario testing, being meaningless.

In the rest of this paper, we will concentrate on more positive results. Having chosen an appropriate functional form for a given case study the next question should be that of existence and uniqueness of a stable fixed point. We present results for the linear and sigmoidal cases stating under which settings a unique fixed point is guaranteed. In particular we highlight that there is an upper bound on the parameter λ in the sigmoid function below which a unique fixed point, which is stable, is guaranteed to exist. These results have been applied to enable the analysis of a real world problem. In Section 4 we present a brief summary of a case study concerning bio-based energy production, a full report of this case study is given in [23].

2. Fixed points for the linear FCM

In the linear case fixed points of (1) satisfy

$$\underline{x} = A\underline{x}, \tag{2}$$

that is, they are the eigenvectors of A associated with unit eigenvalues. Note that it is impossible for a fixed point to be unique, due to the multiplicative invariance of (2) (i.e. if $\underline{x}$ is a solution, then so is $a\underline{x}$ for any $a \in \mathbb{R}$). However, all of these fixed points will have the same ordering of the concepts.

If there are two solutions of (2) (v^1 and v^2) with different orderings of the concepts (A has two unit eigenvalues) then, by the superposition principle, there is in fact a two dimensional manifold of fixed points spanned by v^1 and v^2, and $\underline{x} = a_1\underline{v}^1 + a_2\underline{v}^2$ is a solution of (2) for any $a_1, a_2 \in \mathbb{R}$. Within this manifold, the fixed point evolved to (if stable) is determined by the initial conditions, i.e. the vector $\underline{x}^0$. So changing the initial conditions in such a scenario may result in a different ordering of the concepts at the fixed point.

So the dimension of the manifold of fixed points is equal to the number of unit eigenvalues of A. A unit eigenvalue exists when $A - I$ is not invertible. Consider the situation where concept i is a driver, then row i of A only has one entry, a 1 on the diagonal. This means that row i of $A - I$ is empty and thus $A - I$ is not invertible and A has a unit eigenvalue. In the situation where there are m drivers there will be m unit eigenvalues. If there are more than m unit eigenvalues then a small perturbation of the non-driver elements (e.g. changing the strength of all links by 0.00001) would destroy these extra unit eigenvalues. Thus generically there will be exactly m unit eigenvalues.

If concept i is a driver then the ith equation for the eigenvector $\underline{v}^i$ associated with the unit eigenvalue is $\underline{v}^i_i = \underline{v}^i_i$. This freedom means that not only are all the eigenvectors associated with unit eigenvalues independent, but so are the shorter vectors, made from the eigenvectors by extracting the components associated with all the drivers. For instance if concept 1, 3 and 4 were drivers then the eigenvectors associated with the unit eigenvalues are $\underline{v}^1$,$\underline{v}^3$ and $\underline{v}^4$. These three vectors are linearly independent and span the eigenspace associated with the unit eigenvalue. However, the vectors $(\underline{v}^1_1, \underline{v}^1_3, \underline{v}^1_4)$, $(\underline{v}^3_1, \underline{v}^3_3, \underline{v}^3_4)$ and $(\underline{v}^4_1, \underline{v}^4_3, \underline{v}^4_4)$ are also linearly independent.

For any driver i the iterative map (1) (linear) for the value of the concept is $x_i^{n+1} = x_i^n = x_i^0$. That is, for a given initial condition, the values of the drivers at the fixed point it evolves to are the same as their initial values. Thus if there are m drivers then at the fixed point we have m equations

$$x_i^0 = a_1 v_i^1 + \ldots + a_m v_i^m$$

for the m constants a_j. As the shorter vectors made from the eigenvectors v_i by extracting the components associated with the drivers are linearly independent these equations are uniquely solvable for a_j, $j = 1, \ldots, m$. This means that the fixed point (if stable) that the map converges to is independent of the initial value of all nodes which are not drivers. Also, we can multiply the initial value of all the drivers by a constant and the fixed point (if stable) that the map converges to will preserve the ordering of the concepts, since the equations for a_j are linear.

Specifically, we have proven the following theorem.

Theorem 2.1. *Consider the linear FCM, $\underline{x}_{n+1} = A\underline{x}_n$ and let all drivers start with the same initial value. Then (if a stable fixed points exists) the ordering of the concepts at the fixed point the map iterates to will be unique; independent of the initial values of the concepts which are not drivers.*

It cannot be guaranteed that the fixed points of the linear FCM are linearly stable. However we can say that for a given A the stability of all fixed points will be the same. It will be determined by the eigenvalues of A. If they are all less than (or equal to) 1 then all of the fixed points will be linearly stable, otherwise they will all be unstable. So, if there are m drivers then there is either no stable fixed points, or an m dimensional manifold of linearly stable fixed points and choosing the initial values of the drivers to be the same guarantees a unique ordering of concepts.

If there are drivers then the stability of the manifold of fixed points is determined by A which satisfies

$$A \cong \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$$

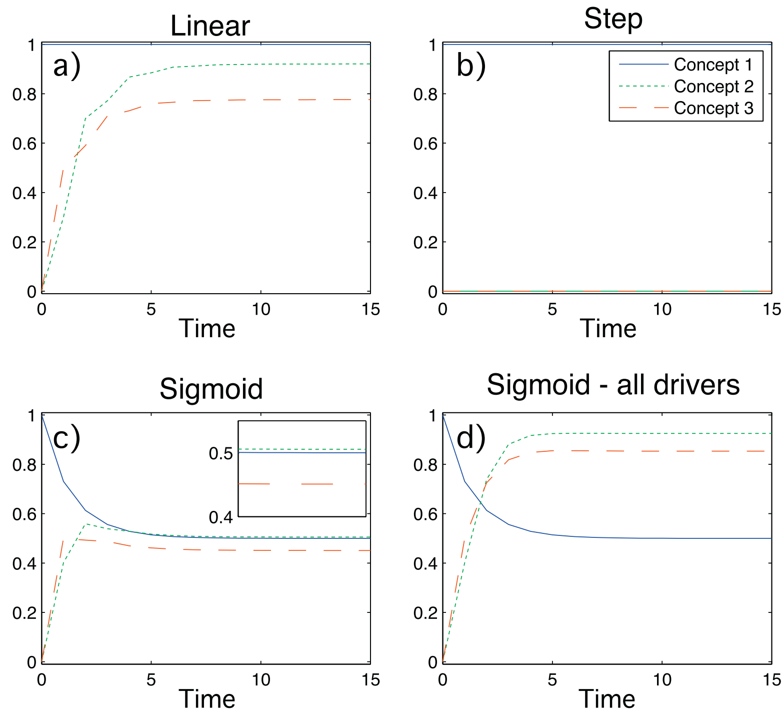

Fig. 2. Forward iterations of (1) with initial condition $x_0 = (1, 0, 0)$ and different functions f. Panel (d) contains the alternate implementation procedure for the sigmoid function (all concepts are drivers). The inset in panel (c) shows a blow up of the fixed point. For the step FCM, (b) concepts 2 and 3 are both zero.

where B can be derived from A by removing all rows and columns associated with the drivers. Now, if the maximal row (or column) sum of the modulus of the entries of B is less than 1, then all eigenvalues of B are less than 1 and so the manifold of fixed points is linearly stable. If B does not satisfy this condition, then we cannot conclude the stability of the manifold of fixed points without explicitly calculating the eigenvalues of B. Indeed, if the manifold of fixed points is unstable then one would see the iterations of the linear FCM blowing up.

3. Fixed points for the sigmoidal FCM

Sigmoidal FCMs restrict the values of the concepts (x_i) to a bounded region in $\mathbb{R}^n$. Usually it is used to enforce positivity of the concepts, it also normalises them, most commonly bounding them above by 1 [2,25,29]. In this section we consider the standard sigmoid function which restricts x_i to the interval [0, 1],

$$f(x; \lambda, h) = \frac{1}{1 + e^{-\lambda(x-h)}}, \tag{3}$$

where the parameter $\lambda \geq 0$ represents the steepness of the function and h is a possible offset. So, for instance, if we consider the limit as $\lambda \to \infty$, $f(x; \lambda, h)$ tends to a step function

$$f_\infty(x; h) = \begin{cases} 0, & x < h \\ 1, & x > h \end{cases}. \tag{4}$$

Considering fixed points of the sigmoidal FCM (1) means considering solutions of

$$\underline{x} = f(A\underline{x}; \lambda, h), \tag{5}$$

where A is the fuzzy weighted adjacency matrix. This equation does not generally have a unique solution, contradicting the implicit assumption that is often made when sigmoidal FCMs are used in practice [13,23,25]. In fact the maximal number of solutions that this equation can have depends on the size of λ. This can be seen by considering a simple example. For this example we shall use the

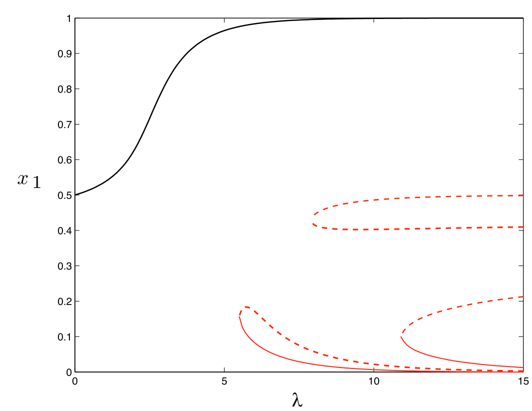

Fig. 3. Changes in the number of fixed points of a sigmoid FCM (1) as λ is varied. Solid lines are stable fixed points while the dashed lines are unstable fixed points.

implementation procedure which says that all concepts are drivers (that is, the diagonal elements of the adjacency matrix are 1).

We take $h = 0.5$ and

$$A = \begin{pmatrix} 1 & 0.2 \\ 0.3 & 1 \end{pmatrix},$$

then as we vary λ the number of solutions of (5) changes, see Fig. 3.

The stable fixed points are shown as solid lines and the unstable ones as dashed lines. For small λ there is a unique solution, which is stable, whilst for larger λ new solutions emerge in pairs via saddle-node/fold bifurcations. For large λ there are three stable fixed points and four unstable ones.

We can generalise this result with the following theorem.

Theorem 3.1. *The number of solutions of (5) depends on the size of λ:*

- *If $\lambda \geq 0$ is small enough then there is a unique solution. This fixed point of the sigmoidal FCM, (1), is linearly stable.*

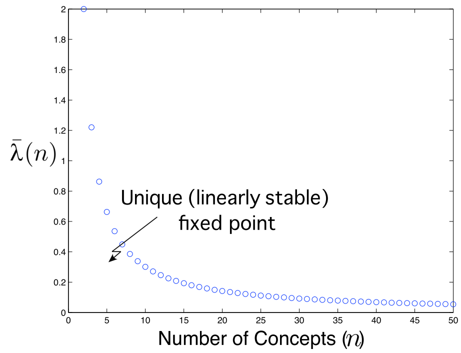

Fig. 4. The minimal value of λ, $\bar{\lambda}(n)$, for which uniqueness of solutions cannot be guaranteed for all A and h.

- *If $\lambda \geq 0$ is large enough there can be multiple (distinct) solutions. In sigmoidal FCMs, (1), many of these fixed points may be linearly stable.*

  If there are m drivers in the system it is possible to construct an $n \times n$ matrix A such that the sigmoidal FCM has at least m (distinct) linearly stable fixed points, for $\lambda \geq 0$ large enough, whose existence and stability are stable to perturbations in A. So for the implementation procedure which takes all concepts as drivers we can construct A so that there are n linearly stable fixed points.

In particular, if we take the limit as $\lambda \to \infty$ then Theorem 3.1 gives that step function FCMs are not guaranteed to have a unique fixed point. In general they may have many stable fixed points.

Proof. See Appendix A. □

Theorem 3.1 gives that the sigmoidal FCM has a unique fixed point (which is stable) if $\lambda > 0$ is small enough. The statement is fairly vague, it would be more useful if we knew exactly what 'small enough' meant. Then we would be able to say whether the values of λ commonly used (1 or 2 – a shifted tanh function) fall within this interval, and, if not make suggestions for replacement values of λ. Kottas et al. [17] derive a similar condition if $\lambda = 1$, for a given A they can say whether the FCM is guaranteed to have a unique fixed point. We aim for something different, to find a bound on λ such that all possible $n \times n$ adjacency matrices, A, are guaranteed to have a unique fixed point.

As a unique solution exists for all A and h with λ small enough we can find an upper bound on λ, which we call $\bar{\lambda}$, such that a unique solution is guaranteed to exist for all A, h and $0 \leq \lambda < \bar{\lambda}$. There is a slight ambiguity in our phrasing here, when we say for all A, we mean for all A of a given size. That is, the concepts are kept the same and we consider all possible ways of joining them up and show that each has a unique fixed point (possibly different) if $0 \leq \lambda < \bar{\lambda}$. That is, we find $\bar{\lambda}(n)$, where n is the size of A – the number of concepts involved in the network, see Fig. 4.

Theorem 3.2. *For $A \in \mathbb{R}^n \times \mathbb{R}^n$ and h given, the sigmoid FCM, (1), has a unique fixed point for all λ such that $0 \leq \lambda < \bar{\lambda}(n)$, this fixed point is stable. $\bar{\lambda}(n)$ satisfies*

$$\left(1 - \frac{\bar{\lambda}(n)}{4}\right)^n - \sum_{i=1}^{n} b_i C_i^n \left(\frac{\bar{\lambda}(n)}{4}\right)^i = 0,$$

where C_i^n are the binomial coefficients, and b_i is given by the recursion relation

$$b_i = ib_{i-1} + (-1)^i, b_0 = 1.$$

$\bar{\lambda}(n)$ is plotted in Fig. 4.

Proof. See Appendix A. □

From this theorem, and the associated plot of $\bar{\lambda}(n)$ we can see that the interval of λ values for which a unique fixed point is guaranteed gets smaller as the number of concepts in the system being modelled grows. One important fact we highlight is the requirement on the size of the problem needed to guarantee that the sigmoid FCM has a unique fixed point when $\lambda = 1$ or $\lambda = 2$ (two common choices for λ). If there are two or more concepts then $\lambda = 2$ will not guarantee a unique fixed point, whilst if there are four or more concepts $\lambda = 1$ will not guarantee a unique fixed point. However, if we choose $\lambda = 0.05$, say, (as done in [23]) then we are guaranteed that the sigmoidal FCM will converge to a unique fixed point (as it is stable) independent of the initial conditions used.

4. Application: the Humber region

The theorems derived in this paper were developed in response to an on-going case study of the Humber region in the UK in which fuzzy cognitive mapping was employed to develop an understanding of a shift towards a bio-based economy. A full report of this case study including details of the participatory work that was undertaken, and a thorough analysis of the results found, is given in [23]. Here we highlight the importance of the theorems described in this paper to that case study with a brief summary of the analysis of one of the FCMs created by stakeholders. For further analysis of the model, including refinements made and further stages of participatory work see [23].

The Humber region is a large, active and diverse industrial area with industries ranging from food processing to oil refining and chemical production. The area provides a substantial proportion of national gas landing and oil refining capacity, as well as being a net energy exporter due to a large number of coal fired power stations and heavy industrial facilities. This region is actively attempting to transition towards a low carbon economy, this transition presents new challenges and opportunities. Currently a shift is being made away from the fossil fuel based economy towards a bio-based economy. In this context our case study of the development of the bio-based economy of the region aims to address factors important to key policy issues. We note here that the industrial aspects of this region cannot be treated in isolation, there are other aspects of the region which must also be considered for a full understanding of the system, for instance the geography of the area means that the major work force must cross a toll paying bridge to reach the industrial complex. Added to this, the river estuary sites most suitable for further ports, renewables and other industrial developments are part of an area of national and international biodiversity and conservation importance. Further, all of this exists in the context of a complicated regulatory and planning environment, which will have a significant impact on how the system will actually develop.

Due to the intricacies of the region we choose to use a participatory methodology to gain insight from the understanding and knowledge of those working and making decisions there. Not only does such participatory work help our understanding of the region, but it also provides the opportunity for the stakeholders present to formalise their understanding of the region and gain new insights. Working with a group of eleven stakeholders we sought to understand key factors or concepts present in their understanding of the region, and how they are interconnected. This means that the methodology of fuzzy cognitive mapping was well suited to our needs. Following the methodology for creating a participatory FCM summarised in the introduction (for a more detailed description of the specific process used see [23]) the participants created a model based around bio-based energy production comprising 16 key concepts and 27 directed fuzzy weighted connections (e.g. weak negative, medium negative, strong positive, etc.). The fuzzy

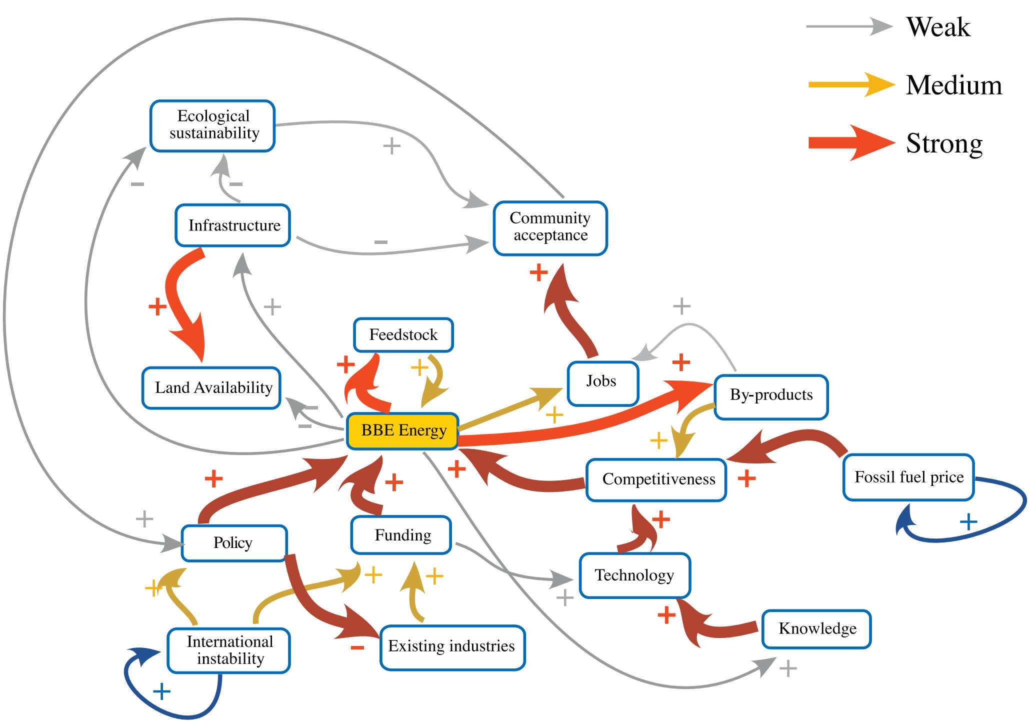

Fig. 5. Humber region bio-based economy FCM from first workshop. Thickness of the links denotes the strength of the influence (caption and figure reproduced from [23]).

cognitive map they produced is shown in Fig. 5 (reproduced from [23]).

This map shows the factors in the region which the stakeholders consider to be important. However it would be useful (for influencing policy decisions for instance) to be able to say which of these key factors are the most important, or the most amenable to effective intervention. If we use the fuzzy cognitive map as an iterative process, i.e. (1), then we can use the values of the concepts at a fixed point to rank those concepts [9,20,23,24]. However, for this ranking to be unique (and not dependent on initial conditions) we require that the iterative system has a unique stable fixed point. We will use Theorems 2.1 and 3.2 to guarantee that this is the case.

We use two different iterative maps, a linear one and a sigmoidal one, as in such a subjective setting we were not able to get the specifics of the update function *f*. The use of the linear one is justified by the fact that it can be easily explained to participants with diverse backgrounds, and that it represents their intuitive understanding. However it does possess some problematic properties, namely the values of concepts can become negative, at which point the fact that a concept with negative strength and a negative impact has a positive effect needs to be carefully justified. To avoid this, sigmoid FCMs are commonly used to bound the values of the concepts between 0 and 1. As we pointed out in our previous example (Section 1.1) using the linear and sigmoidal FCMs may lead to different rankings of the concepts. So we shall use the similarities between the two rankings we derive to determine which concepts are the most important and which are the least important to bio-based energy production in the Humber region.

From Theorem 2.1 for linear FCMs, making sure that the two drivers (fossil fuel price and international instability) start with the same initial value gives a unique ranking at the fixed point, irrespective of any other initial conditions. Likewise, as there are 16 concepts we can see from Fig. 4 that for the sigmoidal FCM there is a unique ordering of concepts at a fixed point for λ small enough, $\lambda = 0.05$ say. Thus we can have confidence in using the values of the concepts at the fixed point as a ranking of their importance, see Fig. 6 (reproduced from [23]).

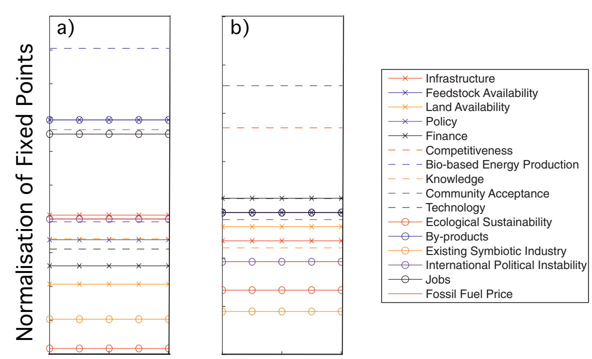

Fig. 6. Fixed point of: (a) linear FCM; (b) sigmoidal FCM.

From Fig. 6 there are five concepts which are in the top seven in both of the rankings (that from the linear FCM, and that from the sigmoidal FCM) and five which are consistently in the bottom eight. We say that these are the five most important concepts, and the five least important concepts respectively, to the development of a bio-based economy. The five factors in the top seven are bio-based energy production, by-products, feedstock availability, competitiveness and jobs. Similarly from our analysis, five of the least important of the stakeholders key concepts are ecological sustainability, existing symbiotic industries, land availability, knowledge and community acceptance. This information could be useful in policy making, or in the development of a more in-depth model of the shift to a bio-based economy. For a more detailed analysis of the implications of the Humber region FCM see [23].

5. Conclusion

In this paper we have investigated fuzzy cognitive mapping and in particular how it is commonly used as a participatory modelling methodology. Specifically, we have studied the fixed points of various different types of fuzzy cognitive map, primarily focusing on the linear FCM and the sigmoid FCM. However, our results also have relevance to the step function FCM. Before proceeding we would like to reiterate the point made in the introduction about the usefulness of the initial stages of the participatory FCM methodology as a way of engaging with stakeholders and as a tool to help frame deeper research into the system under investigation.

Having said that, we draw attention to the issues surrounding the choice of function f and the process by which a static model of a system is turned into a dynamic model by making it an iterative map. We have seen that the choice of function and the way the map is implemented (i.e. which concepts are made drivers) may drastically affect the output of the model, that is, the ordering of the concepts at fixed points. So, in order for the iterative map to give a valid output for the system, the choice of f and of the implementation procedure must be carefully justified. If an f and an implementation procedure can be suitably justified then the output, in terms of orderings, from the iterative map (1) may prove useful in understanding the system and in particular how it responds to different scenarios. If the specific form of f cannot be readily justified then another option may be to use a form of sensitivity analysis based on the rankings from using a variety of different function types. This was done in the application that we presented in Section 4 by combining the results from a linear and a sigmoid FCM to conclude which concepts were the most and least important. A more thorough sensitivity analysis and a discussion of its merits for this application can be found in [23].

Given the ‘fuzzy’ nature of the original static model and the fact that initial conditions cannot be defined ‘precisely’, proceeding with the iterative model only makes sense if a unique fixed point exists. Whether or not this holds will depend on the exact form of f and A, and on the implementation procedure. Here we have studied two cases.

- If f is linear and we use the standard implementation procedure (drivers), then, if a stable fixed point exists and we start all drivers with the same initial value there is a unique ordering at the fixed point the map iterates to. That is, a change in any of the initial conditions for non-driver concepts will not affect the ordering at the fixed point. Note, this does not guarantee the existence of a stable fixed point.
- If f is sigmoidal and we use either implementation procedure (drivers derived from the system, or all concepts are drivers) then a unique (linearly stable) fixed point is guaranteed to exist for $\lambda < \bar{\lambda}(n)$, see Fig. 4. If $\lambda > \bar{\lambda}(n)$ then there may exist multiple stable fixed points. Note that if f is a step function this is equivalent to $\lambda \to \infty$, thus $\lambda > \bar{\lambda}(n)$ means we can never guarantee the uniqueness of fixed points.

Our investigation of the mathematical side of FCMs, and the theorems we have proved, should allow practitioners of fuzzy cognitive mapping to choose initial conditions (linear case), or parameters (sigmoidal case) in such a way that a unique outcome of the iterative modelling procedure can be guaranteed. This will be particularly useful within a participatory context, as is demonstrated by the case study we summarise.

There is still much further work that must be done if this methodology is to be rigorously justified. For instance, there are other possible forms of f to investigate. Also the ordering of concepts at the fixed points may be dependent on the exact value of parameters in the function f. In which case these parameters may also need to be justified in the modelling procedure. Another avenue of further work would be to consider whether all of the links should use the same form for f or whether this form should be specified for each link individually. We plan to do further work in this area, integrating possible improvements to the methodology of fuzzy cognitive mapping into our ongoing study of the bio-based economy of the Humber region [23].

Acknowledgements

We gratefully acknowledge the partial support of the UK Engineering and Physical Sciences Research Council for programme grant EP/H021450/1 (Evolution and Resilience of Industrial Ecosystems ERIE).

Appendix A.

This appendix contains the proofs of Theorems 3.1 and 3.2 which were omitted in the body of the text. Both of these theorem consider the sigmoid FCM. That is $\underline{x}_{n+1} = f(A\underline{x}_n; \lambda, h)$ where

$$f(x; \lambda, h) = \frac{1}{1 + e^{-\lambda(x-h)}}.$$

Proof of Theorem 3.1.

The proof of this theorem is split into two parts. The first considers λ small and the second considers λ large. In particular we shall look at $\lambda = 0$ and $\lim_{\lambda\to\infty}$ and show the results hold for these values of λ. A Taylor expansion (along with the implicit function theorem) is then used to conclude that the result holds for λ ‘close’ to these values.

- Let $\lambda = \epsilon$ where ϵ is a small positive number. If $\epsilon = 0$ then $f(A\underline{x}; 0, h) = \frac{1}{1+e^0} = \frac{1}{2}$ for all $\underline{x}$. Thus the only fixed point of the sigmoidal FCM (the only solution of (5)) is $x_i = \frac{1}{2}$ for all i (every component of $\underline{x}$ is $\frac{1}{2}$). We write this $\underline{x} = \frac{1}{2}\underline{1}$.

  We now let $G(\underline{x}; \epsilon) = \underline{x} - f(A\underline{x}; \epsilon, h)$ for some fixed h. The implicit function theorem says that if $DG_{\underline{x}}(1/2\,\underline{1}; 0)$ is invertible then there exists a unique function $\underline{x}(\epsilon)$ such that $G(\underline{x}(\epsilon); \epsilon) = 0$ and $\underline{x}(0) = 1/2\,\underline{1}$.

  Now, $\partial f((A\underline{x})_i; \epsilon, h)/\partial x_j|_{\epsilon=0} = 0$ for all i and j so $DG_{\underline{x}}(1/2\,\underline{1}; 0) = I$, where I is the identity matrix. This is invertible, which proves that for ϵ, and hence λ, small enough the sigmoid FCM has a unique solution.

  To consider the linear stability of this fixed point we look at the Jacobian of the map. That is $J = D_{\underline{x}}f(A\underline{x}; \epsilon, h)$. First off, note the general relation at a fixed point, $x = f(x; \lambda, h)$,

  $$f'(x; \lambda, h) = \frac{\lambda e^{-\lambda(x-h)}}{\left(1 + e^{-\lambda(x-h)}\right)^2} = \lambda\frac{1-x}{x}x^2 = \lambda x(1-x).$$

  Thus generically at a fixed point of the sigmoidal FCM

  $$J = \lambda A \mathrm{diag}(x_i[1 - x_i]),$$

  where $\mathrm{diag}(x_i[1 - x_i])$ is the diagonal matrix with diagonal entries $x_i[1 - x_i]$. For $\lambda = \epsilon$ the fixed point is $x_i = 1/2 + O(\epsilon)$ for all i, which means that $J = \epsilon/4\,A + O(\epsilon^3)$.

  As A is bounded, we can bound J as

  $$J_{ij} < k\epsilon \quad \forall i, j,$$

  for some $k \in \mathbb{R}$. As A is an $n \times n$ matrix this means that

  $$(J^2)_{ij} < n(k\epsilon)^2 \quad \forall i, j.$$

Similarly

$$(J^m)_{ij} < \frac{1}{n}(nk\epsilon)^m \quad \forall i, j.$$

So $\lim_{m\to\infty} J^m = 0$, which implies the spectral radius of J is strictly less than 1, thus proving the stability of the fixed point. Thus for any A and h, and $\lambda > 0$ small enough the sigmoidal FCM has a unique fixed point, which is linearly stable.

- Let $\lambda = 1/\epsilon$ where ϵ is a small positive number. If $\epsilon = 0$ then $f(x; \lambda, h) \to f_\infty(x; h)$, a step function (4). We construct some fixed points of this step function FCM, $\underline{x} = f_\infty(A\underline{x}; h)$. If $h \neq 0$ then $\underline{x} = \underline{0}$ is a fixed point. For all h, $\underline{x} = \underline{e}_i$ (the standard basis vector) is a solution provided $A_{ii} > h$ and $A_{ji} < h$ for all $j \neq i$. Note that if we choose A such that the diagonal elements are all greater than h and all other elements are less than h then we will have at least n solutions ($n+1$ if $h \neq 0$). (This is easy to achieve if we use the implementation procedure which makes all diagonal elements unitary (drivers)[8,21,26]. If we use the alternative procedure then we can easily choose h so that we have at least as many solutions as we do drivers. Note that if there are no drivers it is still possible to have multiple fixed points, which are the sum of two of the standard basis vectors, however, we ignore this case as we simply wish to show that for a generic A there *may* be more than one fixed point.) From now on we will assume that there are 2 or more drivers.

With ϵ small but non zero. We show that there exists $\underline{\eta}$, $\frac{\eta_j}{\epsilon} << 1$ such that $\underline{x} = \underline{e}_i + \underline{\eta}$ solves (5). Note that by symmetry this will imply that there are at least m distinct solutions, where m is the number of drivers (we ignore the $\underline{x} = \underline{0}$ solution, although this can likewise be expanded). If $\underline{x}$ solves (5) then for each component of $\underline{x}$

$$\underline{e}_{i_j} + \underline{\eta}_j = \frac{1}{1 + e^{-\frac{1}{\epsilon}(A_{ji} - h + k\eta)}},$$

for some constant k, where $\eta = \max_j\{\underline{\eta}_j\}$.

For $j = i$, by our choice of A, $A_{ii} - h > 0$ so $e^{-1/\epsilon(A_{ii}-h)}$ is small. So

$$1 + \underline{\eta}_i = \frac{1}{1 + e^{-\frac{1}{\epsilon}(A_{ii}-h)} - \frac{k\eta}{\epsilon} e^{-\frac{1}{\epsilon}(A_{ii}-h)} + h.o.t.}.$$

Now, η/ϵ is small (by assumption) so

$$\underline{\eta}_i = -e^{-\frac{1}{\epsilon}(A_{ii}-h)} + O\left(\frac{\eta e^{-\frac{1}{\epsilon}}}{\epsilon}\right).$$

Note, this means $\underline{\eta}_i/\epsilon = O(e^{-(1/\epsilon)}/\epsilon)$ which is small for small ϵ. This is consistent with our assumption on η.

For $j \neq i$, $A_{ji} - h < 0$, thus $\frac{1}{e^{-\frac{1}{\epsilon}(A_{ii}-h)}}$ is small. A Taylor expansion in η gives

$$0 + \underline{\eta}_j = \frac{1}{1 + e^{-\frac{1}{\epsilon}(A_{ii}-h)} - \frac{k\eta}{\epsilon} e^{-\frac{1}{\epsilon}(A_{ii}-h)} + h.o.t.}.$$

That is,

$$\underline{\eta}_j = \frac{1}{1 + e^{-\frac{1}{\epsilon}(A_{ji}-h)}} + O\left(\frac{\eta}{\epsilon}\frac{e^{\frac{1}{\epsilon}}}{(1 + e^{\frac{1}{\epsilon}})^2}\right),$$

note that to leading order, this order term is $\eta/(\epsilon e^{1/\epsilon})$, which is small as η/ϵ is small. Thus $\frac{\eta_j}{\epsilon} = O\left(\frac{e^{-\frac{1}{\epsilon}}}{\epsilon}\right)$ which is small for small ϵ and means that

$$\frac{\eta}{\epsilon} = O\left(\frac{e^{-\frac{1}{\epsilon}}}{\epsilon}\right)$$

is small and η thus satisfies our assumptions. Thus there exists $\underline{\eta}$ small such that $\underline{x} = \underline{e}_i + \underline{\eta}$ is a solution of (5). This is true for all drivers i (by construction of A), thus for $\lambda = 1/\epsilon$ large there exists at least m distinct solutions.

We prove the stability of these solutions in the same manner as the λ small case. Recall that the Jacobian at the fixed point is given by

$$J = \lambda \mathrm{diag}(x_i[1 - x_i])A.$$

If (for a specific fixed point) we write $\tilde{\underline{\eta}} = (|\underline{\eta}_1|, |\underline{\eta}_2|, \ldots, |\underline{\eta}_n|)$ then $x_i(1 - x_i) = \tilde{\underline{\eta}}_i + O(\eta^2)$ which means that

$$J \leq \frac{k\eta}{\epsilon}A,$$

for some $k \in \mathbb{R}$.

As $\frac{\eta}{\epsilon}$ is small $\lim_{m\to\infty} J^m = 0$. Thus the spectral radius is less than one and the fixed point is stable. By extension all m of the fixed points of the form $\underline{x} = \underline{e}_i + \underline{\eta}$ are stable.

□

To prove Theorem 3.2 we first of all require some relations for a determinant like object, $det_{||}$. The difference between this object and the determinant is that every negative is changed in to a positive at each stage of its calculation. We define $det_{||}$ iteratively: If

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

then

$$det_{||}(A) = |a_{11}a_{22}| + |a_{21}a_{12}|.$$

If $A \in \mathbb{R}^n \times \mathbb{R}^n$ then the *minor* M_{1j} is created by removing the first row of A and the jth column. Then we define

$$det_{||}(A) = \sum_{j=1}^{n} |a_{1j} det_{||}(M_{1j})| \tag{6}$$

With this definition, we have the following lemma.

Lemma A.1. Take α and $\beta \in \mathbb{R}$, we can define a sequence of matrices X^n and $Y^n \in \mathbb{R}^n \times \mathbb{R}^n$ for $n \geq 2$ as follows: The diagonal entries of X^n are α and all other entries are β, Y^n is the same as X^n with the exception that $Y^n_{11} = \beta$. Then

$$det_{||}(X^{n+1}) = |\alpha det_{||}(X^n)| + n|\beta det_{||}(Y^n)| \quad \text{and}$$
$$det_{||}(Y^{n+1}) = |\beta det_{||}(X^n)| + n|\beta det_{||}(Y^n)|.$$

The operation $det_{||}(X^n)$ will produce a polynomial in $|\alpha|^{n-i}|\beta|^i$ the coefficients of this polynomial can be written as

$$b_i C^n_i \quad \text{where} \quad b_i = ib_{i-1} + (-1)^i \text{ and } b_0 = 1.$$

Note that this is the same b_i as that used in Theorem 3.2.

Proof. Proving that $det_{||}(X^{n+1}) = |\alpha det_{||}(X^n)| + n|\beta det_{||}(Y^n)|$ is straightforward as $det_{||}$ is defined recursively. Consider the minors of X^{n+1}. Due to the form of X, $M_{11} = X^n$ and M_{1j} for $j = 2, \ldots, n+1$ are Y^n, possibly with some of the rows swapped. As $det_{||}(A)$ is invariant to swapping rows of A, the definition (6) gives $det_{||}(X^{n+1}) = |\alpha det_{||}(X^n)| + n|\beta det_{||}(Y^n)|$. The recursive relationship for $det_{||}(Y^{n+1})$ is shown similarly.

Now $det_{||}(X^n)$ is a polynomial in $|\alpha|^{n-i}|\beta|^i$, consider the coefficients as a sequence K_i^n for $i = 0, \ldots, n$, and the coefficients of the polynomial $det_{||}(Y^n)$ as the sequence L_i^n for $i = 0, \ldots, n$. We show that $K_0^n = 1$, $K_i^n = b_i C_i^n$, $L_0^n = 0$ and $L_i^n = C_{i-1}^{n-1} \frac{b_{i+1}}{i}$ for all $i \in 1, \ldots, n$ and all $n \geq 2 \in \mathbb{N}$.

Let $n = 2$, then $det_{||}(X^2) = \alpha^2 + \beta^2$, that is the coefficients are (1, 0, 1). So $K_0^2 = 1$, $0 = K_1^2 = b_1 C_1^2 = 0$ (note that $b_1 = 0$, $b_2 = 1$) and $1 = K_2^2 = b_2 C_2^2 = 1 \cdot 1 = 1$. That is $K_0^n = 1$ and $K_i^n = b_i C_i^n$ for $n = 2$. Also, $det_{||}(Y^2) = \alpha\beta + \beta^2$, so the coefficients are (0, 1, 1). That is, $L_0^2 = 0$, $1 = L_1^2 = C_0^1(b_2/1) = 1$ and $1 = L_2^2 = C_1^1(b_3/2) = 1(2/2) = 1$. That is $L_0^n = 0$ and $L_i^n = C_{i-1}^{n-1} \frac{b_{i+1}}{i}$ for $n = 2$.

We assume that this relationship holds for $n = N$ and show it holds for $n = N+1$, then by induction we have shown that it holds for all $n \geq 2$ and in particular have proven the statement in the lemma. For $n = N+1$

$$det_{||}(X^{n+1}) = |\alpha| det_{||}(X^n) + n|\beta| det_{||}(Y^n),$$

so

$$\sum_{i=0}^{N+1} K_i^{N+1} |\alpha|^{N+1-i} |\beta|^i = \sum_{j=0}^{N} K_j^N |\alpha|^{N+1-j} |\beta|^j + NL_j^N |\alpha|^{N-j} |\beta|^{j+1}$$
$$= K_0^N |\alpha|^{N+1} + NL_N^N |\beta|^{N+1} + \sum_{j=1}^{N} (K_j^N + NL_{j-1}^N) |\alpha|^{N+1-j} |\beta|^j.$$

Similarly

$$\sum_{i=0}^{N+1} L_i^{N+1} |\alpha|^{N+1-i} |\beta|^i = \sum_{j=0}^{N} K_j^N |\alpha|^{N-j} |\beta|^{j+1} + NL_j^N |\alpha|^{N-j} |\beta|^{j+1}$$
$$= \sum_{j=1}^{N+1} (K_{j-1}^N + NL_{j-1}^N) |\alpha|^{N+1-j} |\beta|^j.$$

Matching coefficients gives,

$$K_0^{N+1} = K_0^N, K_{N+1}^{N+1} = NL_N^N, K_i^{N+1}$$
$$= K_i^N + NL_{i-1}^N, L_0^{N+1} = 0, L_i^{N+1} = K_{i-1}^N + NL_{i-1}^N.$$

So $K_0^{N+1} = 1$, $K_{N+1}^{N+1} = NL_N^N = N\frac{b_{N+1}}{N} = b_{N+1} = b_{N+1} C_{N+1}^{N+1}$ and

$$K_i^{N+1} = K_i^N + NL_{i-1}^N = b_i C_i^N + NC_{i-2}^{N-1} \frac{b_i}{i-1} = b_i \left[C_i^N + C_{i-1}^N\right] = b_i C_i^{N+1}.$$

Also

$$L_i^{N+1} = K_{i-1}^N + NL_{i-1}^N = b_{i-1} C_{i-1}^N + NC_{i-2}^{N-1} \frac{b_i}{i-1}$$
$$= C_{i-1}^N [b_{i-1} + b_i] = C_{i-1}^N \frac{b_{i+1}}{i}.$$

This last identity is derived from the recursion relation for b_i.

Thus, by induction, we have $K_0^n = 1$, $K_i^n = b_i C_i^n$, $L_0^n = 0$ and $L_i^n = C_{i-1}^{n-1} \frac{b_{i+1}}{i}$ for all $i \in 1, \ldots, n$ and all $n > 2 \in \mathbb{N}$, completing the proof. □

This now allows us to prove Theorem 3.2.

Proof of Theorem 3.2

For any A and h we know that if λ is small enough the sigmoid FCM has a unique fixed point, Theorem 3.1. For there to be multiple fixed points (or no fixed points) a bifurcation needs to occur as we vary λ. For a given size of A, n, we find the smallest value of λ where such a bifurcation can occur. This is $\bar{\lambda}(n)$. For a bifurcation to occur, the Jacobian of the map needs to be non-invertible, i.e. have determinant zero.

Recall (from proof of Theorem 3.1) that the Jacobian of f (at a fixed point) can be written as

$$J = \lambda A \mathrm{diag}(x_i[1 - x_i]).$$

At the fixed point $\underline{x}$, (5) gives

$$\frac{d\underline{x}}{d\lambda} = J\frac{d\underline{x}}{d\lambda} + \frac{\partial f(A\underline{x}; \lambda, h)}{\partial \lambda}.$$

Thus, a bifurcation occurs when $I - J$ is not invertible.

Note that (under either implementation procedure of sigmoidal FCMs) $A_{ii} \in [0, 1]$ whilst $A_{ij} \in [-1, 1]$ for all $i \neq j$. So $A_{ij}A_{kl} \in [-1, 1]$ for all i, j, k and l. We write this in short hand as $A^2_{..} \in [-1, 1]$. Likewise $A^3_{..}$ is the product of any three elements of A and $A^3_{..} \in [-1, 1]$. In fact $A^i_{..} \in [-1, 1]$ for all i. We also define $\chi_i = x_i(1 - x_i)$, as $x_i \in [0, 1]$, $\chi_i \in [0, (1/4)]$.

Consider $n = 2$, then a bifurcation occurs when

$$0 = det(I - J) = (1 - A_{11}\lambda\chi_1)(1 - A_{22}\lambda\chi_2) - \lambda^2\chi_1\chi_2 A_{12}A_{21}. \quad (7)$$

If $\lambda < 4$ then $0 < (1 - (\lambda/4)) \leq (1 - A_{ii}\lambda\chi_i) \leq 1$. Also $\lambda\chi_i A_{..} \in [-(\lambda/4), (\lambda/4)]$. Thus, the right hand side of (7) is in

$$\left[\left(1 - \frac{\lambda}{4}\right)^2 - \left(\frac{\lambda}{4}\right)^2, 1 + \left(\frac{\lambda}{4}\right)^2\right].$$

If $\lambda < 2$ then this interval is strictly positive and so (7) has no solutions (no bifurcation happens). If $\lambda \geq 2$ then 0 is contained within the interval, thus it is possible to solve (7) for some A, and $\underline{x}$, meaning a bifurcation may occur for a given A and h. Note that we use the same bounds for all χ_i, and the same bounds for all A_{ij}. Thus, as long as terms are not cancelled out, we can simply use χ and $A_.$ rather than keeping track of the individual indices. So the lower bound on the right hand side of (7) is the same as a lower bound on

$$(1 - \lambda\chi)^2 - \lambda^2\chi^2 A^2_{..} = \left(1 - \frac{\lambda}{4}\right)^2 + 1 - det_{||}(X^2),$$

where $\alpha = 1$ and $\beta = (\lambda/4)$. $\bar{\lambda}(2)$ is then the value of λ such that this lower bound is zero, i.e.

$$\left(1 - \frac{\lambda}{4}\right)^2 + 1 - det_{||}(X^2) = 0.$$

If we consider $n = 3$ in the same way then the bounds on $det(I - J)$ are given as the bounds on

$$(1 - A_{ii}\lambda\chi)^3 - 3(1 - A_{ii}\lambda\chi)\lambda^2\chi^2 A^2_{..} + 2\lambda^3\chi^3 A^3_{..}.$$

Due to the bounds for χ, A_{ii} and $A_{..}$ this is bounded by,

$$\left[\left(1 - \frac{\lambda}{4}\right)^3 - 3\left(\frac{\lambda}{4}\right)^2 - 2\left(\frac{\lambda}{4}\right)^3, 1 + 3\left(\frac{\lambda}{4}\right)^2 + 2\left(\frac{\lambda}{4}\right)^3\right].$$

Again, this lower bound can be written as

$$\left(1 - \frac{\lambda}{4}\right)^3 + 1 - det_{||}(X^3),$$

where $\alpha = 1$ and $\beta = \lambda/4$. Setting this lower bound to be zero defines $\bar{\lambda}(3)$, below which no bifurcations can occur.

We can follow this process for any n, giving that $\bar{\lambda}(n)$ satisfies

$$0 = \left(1 - \frac{\lambda}{4}\right)^n + 1 - det_{||}(X^n). \quad (8)$$

We explain this as follows. The only term in the determinant of $I - J$ which cannot be negative (for $\lambda < 4$) is the product of the diagonal entries. This is bounded by $(1 - A_{ii}\lambda\chi)^n$ which in turn is bounded below by $(1 - (\lambda/4))^n$. This is the first term in (8). All other terms in $det(I - J)$ can be negative, because of the symmetry in the bounds $A^k_{\cdot\cdot} \in [-1, 1]$ (for all k) the minimal value of these terms is the negative of the maximal value. As the maximal value of $(1 - A_{ii}\lambda\chi_i)$ is 1 and the maximal value of $\lambda A_{\cdot\cdot}\chi$ is $\lambda/4$, the remaining terms of the bound on $det(I - J)$ are a polynomial in $1^{n-i}(\lambda/4)^i$. The coefficients come from the determinant, where all negative signs are ignored, that is, the remaining polynomial is $det_{||}(X^n)$ by definition (minus the first term which would be $1^n = 1$). This gives (8) and by Lemma A.1 completes the proof. □

References

[1] J. Aguilar, Survey about fuzzy cognitive maps, International Journal of Computational Cognition 3 (2) (2005) 27–33.

[2] S. Bueno, JL, Salmeron benchmarking main activation functions in fuzzy cognitive maps, Expert Systems with Applications 36 (3) (2009) 5221–5229.

[3] E.H. Cakmak, H. Dudu, O. Eruygur, M. Ger, S. Onurlu, Ö. Tonguç, Participatory fuzzy cognitive mapping analysis to evaluate the future water in the Seyhan Basin, Journal of Water and Climate Change 4 (2) (2013) 131–145.

[4] C.W. Chen, Delay independent criterion for multiple time-delay systems and its application in building structure control systems, Journal of Vibration and Control 19 (3) (2013) 395–414.

[5] J.P. Craiger, R.J. Weiss, D.F. Goodman, A.A. Butler, Modelling organizational behaviour with fuzzy cognitive maps, Internation Journal of Computational Intelligence and Organisations 1 (1996) 120–133.

[6] S. Fons, G. Achari, T. Ross, A fuzzy cognitive mapping analysis of the impacts of an eco-industrial park, Journal of Intelligent and Fuzzy Systems 15 (2004) 75–88.

[7] W. Froelich, E.I. Papageorigiou, M. Samarinas, K. Skriapas, Application of evolutionary fuzzy cognitive maps to the long-term prediction of prostate cancer, Applied Soft Computing 2 (12) (2012) 3810–3817.

[8] P.P. Groumpos, Fuzzy cognitive maps: basic theories and their applications to complex systems, Studies in Fuzziness and Soft Computing 247 (2010) 1–22.

[9] B.F. Hobbs, S.A. Ludsin, R.L. Knight, P.A. Ryan, J. Bilberhofer, J.J.H. Ciborowki, Fuzzy cognitive mapping as a tool to define management objectives for complex ecosystems, Ecological Applications 12 (5) (2002) 1548–1565.

[10] C.F. Hsu, C.M. Lin, R.G. Yeh, Supervisory adaptive dynamic RBF-based neural-fuzzy control system design for unknown nonlinear systems, Applied Soft Computing 13 (2013) 1620–1626.

[11] A.J. Jetter, R.C. Sperry, Fuzzy cognitive maps for product panning: using stakeholder knowledge to achieve corporate responsibility, in: 46th Hawaii International Conference on System Science (HICSS), 2013, pp. 925–934.

[12] A. Kannappan, A. Tamilarasi, E.I. Papageorgiou, Analyzing the performance of fuzzy cognitive maps with non-linear Hebbian learning algorithm in predicting autistic disorder, Expert Systems and Applications 8 (3) (2011) 1282–1292.

[13] M.S. Khan, M. Quaddus, Group decision support using fuzzy cognitive maps for causal reasoning, Group Decision and Negotiation 13 (5) (2004) 463–480.

[14] K. Kok, The potential of fuzzy cognitive maps for semi-quantitative scenario development, with an example from Brazil, Global Environmental Change 19 (1) (2009) 122–133.

[15] B. Kosko, Fuzzy cognitive maps, International Journal of Man–Machine Studies 24 (1) (1986) 65–75.

[16] T.L. Kottas, Y.S. Boutalis, M.A. Christodoulou, A new method for weight updating in fuzzy cognitive maps using system feedback, ICINCO (2005) 202–209.

[17] T.L. Kottas, Y.S. Boutalis, M.A. Christodoulou, Fuzzy cognitive networks: adaptive network estimation and control paradigms, Fuzzy Cognitive Maps, Studies in Fuzziness and Soft Computing 247 (2010) 89–134.

[18] V.K. Mago, H.K. Morden, C. Fritz, T. Wu, S. Namazi, P. Geranmayeh, R. Chattopadhyay, V. Dabbaghian, Analyzing the impact of social factors on homelessness: a fuzzy cognitive map approach, BMC Medical Informatics and Decision Making 13 (2013).

[19] A. Mas-Colell, A.M. Whinston, J.R. Green, Microeconomic Theory, Oxford University Press, USA, 1995.

[20] G.A. Mendoza, R. Prabhu, Participatory modelling and analysis for sustainable forest management: Overview of soft system dynamics model and applications, Forest Policy and Economics 9 (2006) 179–196.

[21] E.I. Papageorgiou, P.P. Groumpos, A new hybrid method using evolutionary algorithms to train fuzzy cognitive maps, Applied Soft Computing 5 (2005) 409–431.

[22] E.I. Papageorgiou, C.D. Stylios, P.P. Groumpos, Active Hebbian learning algorithm to train fuzzy cognitive maps, International Journal of Approximate Reasoning 37 (2004) 219–249.

[23] A.S. Penn, C.J.K. Knight, D.J.B. Lloyd, D. Avitabile, K. Kok, F. Schiller, A. Woodward, A. Druckman, L. Basson, Participatory development and analysis of a fuzzy cognitive map of the establishment of a bio-based economy in the Humber region, PLoS ONE 8 (11) (2013), http://dx.doi.org/10.1371/journal.pone.0078319.

[24] L.S. Soler, K. Kok, G. Câmara, A. Veldkamp, Using fuzzy cognitive maps to describe current system dynamics and develop land cover scenarios: a case study in the Brazilian Amazon, Journal of Land Use Science 7 (2) (2012) 149–175.

[25] C.D. Stylios, V.C. Georgopoulos, G.A. Malandraki, S. Chouliara, Fuzzy cognitive map architectures for medical decision support systems, Applied Soft Computing 8 (3) (2008) 1243–1251.

[26] C.D. Stylios, P.P. Groumpos, Modelling complex systems using fuzzy cognitive maps, IEEE Transactions on Systems, Man, and Cybernetics, Part A 34 (1) (2004) 155–162.

[27] A.K. Tsadiras, Comparing the inference capabilities of binary, trivalent and sigmoid fuzzy cognitive maps, Information Sciences 178 (20) (2008) 3880–3894.

[28] M. van Vliet, K. Kok, T. Veldkamp, Linking stakeholders and modellers in scenario studies: the use of fuzzy cognitive maps as a communication and learning tool, Futures 42 (1) (2010) 1–14.

[29] G. Xirogiannis, J. Stefanou, M. Glykas, A fuzzy cognitive map approach to support urban design, Expert Systems with Applications 26 (2) (2004) 257–268.

[30] K. Yeh, C.W. Chen, D.C. Lo, K.F.R. Liu, Neural-network fuzzy control for chaotic tuned mass damper systems with time delays, Journal of Vibration Control 8 (6) (2012) 785–795.